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The lines of Electric Force due to a moving electron

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Introduction

The values of the electric and magnetic forces in the electromagnetic field due to a moving point charge, or electron, have been given by Liénard[1], Wiechert[2], Heaviside[3], and, in detail, by Abraham[4]. The expressions are, however, so complicated in the case of perfectly general motion that the form of the lines of force was not obtained except for the simplest case — that of uniform motion in a straight line. The field of force for this case was known[5] long before that for the general case was worked out and it was shewn that the lines of electric force at any time t were straight lines

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1. L'Éclairage électrique 16. (1898) pp. 5, 53, 106.
2. Arch. Néerlandaises (2) Vol. 5. (1900) p. 549
3. Nature 67. (1902) p. 6.
4. Ann. d. Phys. 14. p 236 (1904)
   Theorie der Elektrizität. Band 2 §§ 13 — 15. Leipzig (1914)
5. O. Heaviside. Phil. Mag Vol 27. p 324 (1889)

through the position of the electron at time t, a result which may be expressed by saying that an electron in uniform motion along a straight line convects its field along with it.

The form of the lines of electric force is of interest in connection with a theory, advanced by Sir J. J. Thomson[1], as to the structure of the ether and the nature of Röntgen rays and light. Following Faraday[2] he regards the lines of force not merely as an abstract mathematical concept but as physical realities. He makes, however, one important modification in Faraday's theory. If we have two charges of opposite sign they will be joined by lines of force, the direction of a line at any point being the direction of the electric force at that point. These are the lines of force which Faraday imagined as physical realities endowed with such properties as tension along their length and mutual transverse

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1. Thomson. Proc. Camb. Phil. Soc. 15 p 65 (1909)

2. A full account of Faraday's theory is given by Thomson in his "Recent Researches in Electricity and Magnetism" Ch. 1.

repulsion. In Thomson's modification of the theory each charge is supposed to carry its own lines of force with it independently of the presence of other charges. In a field containing many charges there will be many lines of force crossing at any point and these give a resultant electric force at that point whose direction is that of the Faraday line at that point. Accordingly the lines of force attached to a point charge are conceived by Thomson as more fundamental than the idea of the point charge itself and may, in fact, be used to define the point charge

To account for results obtained in experiments on the ionisation of gases by Röntgen rays and ultra-violet light Thomson[1] has modified his theory slightly by supposing that the lines of force are not distributed in a continuous manner round the point charge. (He had previously[2] made this modification in Faraday's concept)

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[1] Phil. Mag. 19. p 301 (1910)

[2] Proc. Camb. Phil. Soc. 14. p 417. (For this and other theories as to the structure of the ether see Bateman. Electrical and Optical Wave Motion. Camb. 1915. Ch. 8

§ 1.

Recently a method was given by Bateman[1] which reduces the problem of finding the equations, at any time t, of the lines of electric force due to a point charge moving in any way to the solution of a differential equation of Riccati's type. It will be shewn that this differential equation can always be integrated if the path of the electron lies wholly in a plane. The following cases, which are those of most interest, of motion in a plane have been worked out in detail —

(a) Rectilinear motion with any velocity and acceleration

(b) Uniform motion in a circle

It has been found possible to integrate the equation in one case of motion in three dimensions — that of uniform motion in a circular helix. Further it has been found that if one solution of the Riccatian equation is known a second can be derived from it and accordingly the general

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[1] Bulletin. Amer. Math. Soc. 2nd Series. Vol 21. No 6. p. 308 (1915)

American Journal. April (1915)

solution is reduced to the evaluation of a single quadrature.

It will be convenient to give here a brief account of the method by which Bateman obtained the Riccatian equation referred to. Let C be the curve along which the electron is moving and let $x = \xi$, $y = \eta$, $z = \zeta$ be the coordinates of the point of C occupied by the electron at time τ so that ξ, η, ζ are functions of τ alone. Denote differentiations with regard to τ by primes so that the velocity of the electron is $\bar{v} = (\xi', \eta', \zeta')$ and its acceleration is $\bar{v}' = (\xi'', \eta'', \zeta'')$ where bars are used to distinguish vector quantities from scalars. Let P be a point in space whose coordinates are x y z. A disturbance emanating from the electron at time τ will reach the point P at time t if $c(t-\tau) = r$, where c is the velocity of light, and $r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ Denote by $\bar{r}$ the vector OP (where O is the position of the electron at time τ) and let (l, m, n) be the direction cosines of $\bar{r}$. Then the electric force at P at time t

is known to be[1]

$$E = \frac{e}{r^2}\left\{\frac{\overline{r}}{r} - \frac{\overline{v}}{c}\right\}\left\{1 - \frac{v^2}{c^2} + \frac{(\overline{v}'\,\overline{r})}{c^2}\right\}\left(\frac{\partial\tau}{\partial t}\right)^3 - \frac{e\,\overline{v}'}{rc^2}\left(\frac{\partial\tau}{\partial t}\right)^2$$

where e is the charge on the electron.

If we differentiate the equation $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$ we get $\frac{\partial r}{\partial\tau} = \lambda - c$ where $\lambda = c - \frac{1}{r}(\overline{r}\;\overline{v})$ and since $r = c(t-\tau)$ we have $c\frac{\partial t}{\partial\tau} = \lambda$. We shall be interested in the direction only of E at P and an easy transformation of the expression for E shows that it has the direction of the vector

$$(c^2 - v^2)\left(\overline{v} - c\frac{\overline{r}}{r}\right) + r\left\{\lambda\,\overline{v}' + \left(\overline{v} - c\frac{\overline{r}}{r}\right)p\right\}$$

where $p = l\xi'' + m\eta'' + n\zeta'' = \frac{1}{r}(\overline{r}\;\overline{v}')$

Let P' be a point consecutive to P and let its coordinates be $x+dx$, $y+dy$, $z+dz$. P' will be reached at time t by a disturbance which has emanated from the electron at time $\tau + d\tau$ say. Let O' be the point of C occupied by the electron at this time $\tau + d\tau$. The x-component of the vector equation

$$\overline{PP'} = \overline{PO} + \overline{OO'} + \overline{O'P'} \quad \text{may be written}$$

$$dx = -rl + d\xi + c(l + dl)(t - \tau - d\tau) \quad \text{or}$$

$$\frac{dx}{d\tau} = +r\frac{dl}{d\tau} + \frac{d\xi}{d\tau} + cl$$

Thus if s is the vector $\overline{PP'}$ we have $\overline{s}' = \left(\overline{v} - c\frac{\overline{r}}{r}\right) + r(l', m', n')$ where $(l'\ m'\ n')$ is used to denote the vector whose components are l', m', n'

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[1] Abraham. loc. cit.

Comparing this with the expression giving the direction of the electric force at P we see that:- PP′ is the direction of the electric force at P if the three equations

$$(c^2 - v^2)\frac{d}{d\tau}(l, m, n) = \lambda(\xi'', \eta'', \zeta'') + p(\xi' - cl, \eta' - cm, \zeta' - cn) \cdots (A)$$

are satisfied. From the manner in which these equations were derived we see that, if (l, m, n) is a solution of these equations satisfying the equation $l^2 + m^2 + n^2 = 1$, the equations $x = \xi + c(t-\tau)l$; $y = \eta + c(t-\tau)m$; $z = \zeta + c(t-\tau)n$ (where τ is a parameter satisfying $\tau < t$) give a line of electric force at time t. It will be noticed that, if at each position of the electron we imagine a particle projected with velocity c in the direction (l, m, n), the aggregate of these particles at time t will form a line of force. All the lines at any time t pass through the position of the electron at that time.

In order to obtain the solutions (l, m, n) which satisfy $l^2 + m^2 + n^2 = 1$ put $\sigma = \dfrac{l + im}{1 + n} = \dfrac{1 - n}{l - im}$ where $i^2 = -1$ After an easy reduction we obtain the Riccatian equation

$$2(c^2 - v^2)\frac{d\sigma}{d\tau} = \phi(c - \zeta') + \phi'\zeta'' - 2\sigma\{c\zeta'' + i(\xi'\eta'' - \xi''\eta')\} + \sigma^2\{\psi'\zeta'' - (c + \zeta')\psi''\}$$

where $\phi = \xi + i\eta$; $\psi = \xi - i\eta$

If σ is a solution of this equation and if s is its conjugate

$$l = \frac{\sigma+s}{1+\sigma s}\ ; \quad m = -i\,\frac{\sigma-s}{1+\sigma s}\ ; \quad n = \frac{1-\sigma s}{1+\sigma s}$$

is a solution of the equations (A) which satisfies $l^2+m^2+n^2=1$.

It is interesting to note that the equations

$$(c^2-v^2)\frac{d}{d\tau}(l,\ m,\ n) = \lambda(\xi'',\ \eta'',\ \zeta'') + p(\xi'-cl,\ \eta'-cm,\ \zeta'-cn)$$

are a generalisation of the equations

$$\frac{d}{d\tau}(l,\ m,\ n) = (mr-nq,\quad np-lr,\quad lq-mp) \quad \text{-----} \quad (B)$$

to which they reduce on making $c=0$ and $v^2(p,q,r) = [\bar{v}'\ \bar{v}]$ the square brackets being used to denote a vector product. These equations occur in the kinematics of a rigid body and have been considered at length by Darboux[1]. Several writers[2] have extended these equations to the case of four or more variables and have obtained results similar to those obtained by Darboux

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1. Théorie des Surfaces. I. Ch's 2 and 3.

2. Craig, Cole, and Eiesland; all in Amer. Journal Vol 20 (1898)
   Hatzidakis Amer. Journal. Vol 23 (1901)
   Eiesland Amer. Journal Vol 28 (1906)
   Matsumoto. Memoirs of Coll. of Sci. Kyoto Imperial Univ. Vol I. No 7. Jan 1916.

§ 2.

Associated directions of projection.

A solution (l, m, n) of the equations

$$\mu \frac{d}{d\tau}(l, \ m, \ n) = \lambda(\xi'', \eta'', \zeta'') + p(\xi' - cl, \eta' - cm, \zeta' - cn)$$

which have been mentioned in the preceding paragraph will be called a *direction of projection* provided $l^2 + m^2 + n^2 = 1$. It will now be shewn that to every direction of projection there corresponds, in an involutory manner, a second direction which shall be called the *associated direction of projection.*

Let O be the position at time τ of the moving electron, so that the Cartesian coordinates of O are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$. Let OP be a direction of projection at O so that the direction cosines of OP are (l, m, n) where (l, m, n) are a solution of the fundamental equations (A). Let Q be the position which would have been occupied by the electron at a time $t > \tau$ if it had continued to move with velocity $v = (\xi', \eta', \zeta')$ along the tangent

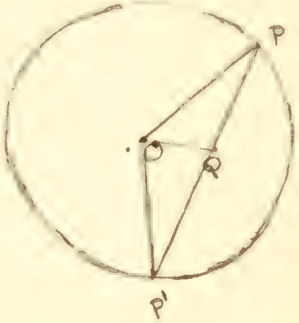
to the path at O. Thus the coordinates of Q are

$$(x_1, \; y_1, \; z_1) = \left(\xi + \xi'(t-\tau), \; \eta + \eta'(t-\tau), \; \zeta + \zeta'(t-\tau)\right)$$

If we make $OP = c(t-\tau)$ and describe a sphere with centre O and radius OP, then PQ will meet the sphere again in a point P'. OP' is the associated direction of projection at O

To prove this write $c(t-\tau) = r$ so that, if the coordinates of P are $(x_0 \; y_0 \; z_0)$,

$$(x_0 \; y_0 \; z_0) = (\xi + rl, \; \eta + rm, \; \zeta + rn)$$



The equations to the straight line PQ are

$$\frac{x - x_1}{x_1 - x_0} = \frac{y - y_1}{y_1 - y_0} = \frac{z - z_1}{z_1 - z_0}$$

Putting each of these fractions equal to θ, say, we get three equations of the type $x - \xi = \xi'(t-\tau) + \theta(\xi' - cl)(t-\tau)$; (for $x_1 - x_0 = (t-\tau)(\xi' - cl)$ etc)

Let (X, Y, Z) be the coordinates of P'; then, since P' is on the sphere, $(X-\xi)^2 + (Y-\eta)^2 + (Z-\zeta)^2 = c^2(t-\tau)^2$. Also P' is on line PQ so that $X - \xi = (t-\tau)\{\xi' + \theta_1(\xi' - cl)\}$ where θ_1 is the value of the parameter θ giving the point P'.

Hence $\theta_1^2\left\{v^2+c^2-2\frac{c}{r}(\bar{r}\;\bar{v})\right\}+2\theta_1\left\{v^2-\frac{c}{r}(\bar{r}\;\bar{v})\right\}+v^2-c^2=0$ where a bar placed over a letter signifies ~~the~~ a vector quantity.

The roots of this quadratic are $\theta_1=-1$ and $\theta_1 = (c^2-v^2)/c^2+v^2-2\frac{c}{r}(\bar{r}\;\bar{v})$. The root $\theta_1=-1$ is the parameter of the point P itself so the second root is the parameter of the point P'. Thus if (L, M, N) are the direction cosines of OP', so that $X-\xi = c(t-\tau)L$ etc. we have, on substituting the value of θ_1 given above in the three equations of the type $X-\xi = (t-\tau)\{\xi'+\theta_1(\xi'-cl)\}$, etc three equations giving L, M, N. These three equations may be written together in the form

$$(\xi'-cL,\ \eta'-cM,\ \zeta'-cN) = h(v^2-c^2)(\xi'-cl,\ \eta'-cm,\ \zeta'-cn)$$

where $h^{-1} = c^2-2\frac{c}{r}(\bar{r}\;\bar{v})+v^2$

If, as in §1 we use the abbreviations

$$\lambda = c-\frac{1}{r}(\bar{r}\;\bar{v})\ ;\quad \mu = c^2-v^2$$

then $h^{-1} = 2c\lambda-\mu$.

Since $\bar{r}$ denotes the vector OP we write $\bar{R}$ for the vector OP' and Γ for the expression $c-\frac{1}{r}(\bar{R}\;\bar{v})$: Then, from the

values obtained for $(L\ M\ N)$ we obtain the equation

$$\underline{\Gamma = h\lambda\mu.}$$

In order to prove that OP' is an associated direction of projection we have merely to shew that (L, M, N) satisfy the equations (A) of § 1. In other words it must be shewn that the three equations

$$\mu \frac{d}{d\tau}(L,\ M,\ N) = \Gamma(\xi',\ \eta'',\ \zeta'') + P(\xi' - cL,\ \eta' - cM,\ \zeta' - cN)$$

are true, where $P = L\xi'' + M\eta'' + N\zeta'' = \frac{1}{r}(\bar{R}\ \bar{v}')$ where $\bar{v}'$ is the vector acceleration of the electron at the point O

If we multiply the three equations (A) of § 1 by ξ', η', ζ' respectively and add they give

$$\mu(p+q) = \lambda\{cp + (\bar{v}\,\bar{v}')\}$$ where we have written q for $l'\xi' + m'\eta' + n'\zeta'$.

This equation is equivalent to $\underline{-\mu\lambda' = \lambda\,[cp + (\bar{v}\,\bar{v}')]}$

Again $\xi' - cL = -\mu h(\xi' - cl)$ where $h^{-1} = 2c\lambda - \mu$. We want to obtain L', so differentiate this equation and it gives

$$cL' - \xi'' = \mu h(\xi'' - cl') - 2h(\bar{v}\,\bar{v}')(\xi' - cl) - 2\mu h^2(\xi' - cl)[c\lambda' + (\bar{v}\,\bar{v}')]$$

On combining the first ~~three~~ two terms on the right hand side of this equation with the term $-\zeta''$ on the left hand side we obtain, after substituting for ℓ' its value from the equations (A) of §1 and for λ' its value just given,

$$cL' = h\left\{c\lambda\zeta'' - (\zeta' - c\ell)\left[cp + 2(\bar{v}\,\bar{v}')\right]\right\} + 2h^2(\zeta' - c\ell)\left\{c^2\lambda p + (c\lambda - \mu)(\bar{v}\,\bar{v}')\right\}$$

or

$$\underline{L' = h\left\{\lambda\zeta'' - (\zeta' - c\ell)p\right\} + 2h^2(\zeta' - c\ell)\lambda\left\{cp - (\bar{v}\,\bar{v}')\right\}}$$

Further from the equations given for (L, M, N) we derive

$$cP = (\bar{v}\,\bar{v}') + \mu h\left\{(\bar{v}\,\bar{v}') - cp\right\}$$

or $P = h\left[2\lambda(\bar{v}\bar{v}') - \mu p\right]$ since $h^{-1} = 2c\lambda - \mu$

Substituting these values of L' and P in the equation

$\mu L' = \Gamma\zeta'' + (\zeta' - cL)P$ we see on dividing across by $\mu h p(\zeta' - c\ell)$ that this equation is true if

$1 - 2hc\lambda = -\mu h$ and this is true since $h^{-1} = 2c\lambda - \mu$.

This, then, proves the theorem stated. If we know a particular solution OP = (ℓ, m, n) of the equations A we

obtain a second solution $OP' = (L\ M\ N)$ in the manner described at the beginning of § 2. Darboux[*] has given a similar result for the equations B of § 1.

There is one particular case, when the associated direction coincides with the original direction of projection, in which we cannot obtain by this method a new solution of the differential equations. In this case PP' is a tangent line to the sphere and so $(l\xi' + m\eta' + n\zeta') = v.\frac{c}{v}$ or $\lambda = 0$. As a matter of fact if we put $\lambda = 0$ in the general equations giving L, M, N we see that $L = l$, $M = m$, $N = n$. In this case the velocity v of the electron is greater than c, the velocity of light. From the equation for λ'

$$-\mu\lambda' = \lambda\left[c\,p + (\bar{v}\,\bar{v}')\right]$$

we see that when $\lambda = 0$, λ' also $= 0$.

In the cases of motion for which a solution has actualy been found explicitly it will be shewn that there are

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[*] Théorie des Surfaces Tome 1 p. 23 (1887).

always solutions consistent with $\lambda = 0$.

§ 3. General solution when motion is in one plane

Let θ be the angle between OP and OQ so that $v \cos\theta = l\xi' + m\eta' + n\zeta'$. It will now be shewn that if the associated directions OP, OP′ are distinct and equally inclined to OQ the motion of the election is in one plane and the solution can be found.

OP and OP′ are equally inclined to OQ if $\cos\theta = \frac{v}{c}$ for then PP′ is perpendicular to OQ. Also $\lambda = c - v\cos\theta$ in general and in this particular case this gives $\mu = c\lambda$. Hence $-c\lambda' = 2(\bar{v}\,\bar{v}')$

On substitution in the equation $-\mu\lambda' = \lambda\{cp + (\bar{v}\,\bar{v}')\}$ of § 2 we get $-c\lambda' = cp + (\bar{v}\,\bar{v}')$ or $\underline{cp = (\bar{v}\,\bar{v}')}$

On substitution in the first of the equations A of § 1 the

gives $\mu(\xi'' - c l') = -(\xi' - c l)(\bar{v}\,\bar{v}')$ which is at once integrable giving

$$\xi' - c l = a(c^2 - v^2)^{1/2}$$ where a is a constant of integration. Similarly we obtain the equations

$$\eta' - c m = b(c^2 - v^2)^{1/2} \;;\quad \zeta' - c n = d(c^2 - v^2)^{1/2}$$ where b and d are constants of integration.

On multiplication by ξ', η', ζ' respectively and addition these yield

$$\underline{a\xi' + b\eta' + d\zeta' = 0} \quad (\text{for } v^2 = c^2 - c\lambda)$$

On using ξ'', η'', ζ'' as multipliers we get, since $(\bar{v}\,\bar{v}') = cp$ the equation

$$\underline{a\xi'' + b\eta'' + d\zeta'' = 0}$$

Thus the velocity and acceleration of the electron are in the same plane. Hence

If a solution of the fundamental equations can be found which satisfies the equation $c\cos\theta = v$ the motion of the electron is in one plane

Conversely if the motion is plane it is easy to pick out a solution for which $c \cos \theta = v$. The theorem of associated directions then gives a second solution which also satisfies $c \cos \theta = v$. Knowing these two solutions the integration of the Riccatian equation of §1 is effected by means of a single quadrature.

Take the plane of motion to be $z = 0$. We have to write $\zeta' = 0$ $\zeta'' = 0$ in our equations. Then if $c \cos \theta = v$ $\mu = c\lambda$ and it is evident that $cl = \frac{\xi'}{\cancel{\theta}}$ $cm = \eta'$ satisfy the first two of the three equations. Substituting these values of l and m in $c \cos \theta = v$, they evidently satisfy it.

To determine n, we introduce the condition for a direction of projection ~~and~~ $l^2 + m^2 + n^2 = 1$, giving

$$cn = \pm (c^2 - v^2)^{1/2}.$$

Either of these values of n satisfies the third equation which is, in this case, $\mu n' = -cn(l\xi'' + m\eta'')$.

Hence the solutions which give $c \cos \theta = v$ are

$$cl = \xi' \; ; \quad cm = \eta' \; ; \quad cn = \pm (c^2 - v^2)^{1/2}$$

To obtain the general direction of projection it is necessary to solve the Riccatian equation. In the case of plane motion this is somewhat simplified and becomes

$$2(c^2 - v^2)\frac{d\sigma}{d\tau} = c\,\phi'' - 2i\sigma(\xi'\eta'' - \xi''\eta') - c\,\sigma^2\psi''$$

where $\sigma(1+n) = l + im$;

Substituting for l, m, n the values just given we have two particular solutions of this equation. Denoting these by σ_1 σ_2 we have

$$c(1+\omega)\,\sigma_1 = \phi' \ ; \qquad c(1-\omega)\sigma_2 = \phi'$$

where $c\omega = (c^2 - v^2)^{1/2}$; the verification by direct substitution in the Riccatian equation is immediate.

The general solution of the Riccatian equation is,* therefore,

$$\sigma = \sigma_2 + \frac{\sigma_1 - \sigma_2}{1 + A\,e^{\int(\sigma_1 - \sigma_2)R\,d\tau}}$$

where A is a constant of integration and $2(c^2 - v^2)R = -c\,\psi''$

A is in general complex, since σ is complex.

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* Cohen. Differential Equations p. 175.

Thus, in the case of plane motion, the problem is reduced to the integration $\int(\sigma_1-\sigma_2)R\,d\tau$ or

$$c\int\frac{(\xi''-\iota\eta'')\,d\tau}{(\xi'-\iota\eta')\sqrt{c^2-\xi'^2-\eta'^2}}$$

This integration can always be effected if the scalar quantity v is constant and in this case. the integral is

$$\left[c/(c^2-v^2)^{1/2}\right]\log\psi'$$ so that

$$\sigma = \sigma_2 + \frac{\sigma_1-\sigma_2}{1+A\psi'^{\frac{1}{\omega}}}$$

If $\eta=0$ $\eta'=0$ $\eta''=0$ it is evident that the integration can be carried out. This is the case of rectilinear motion which will be treated in detail. The case of most interest is that of uniform motion in a circle and the solution in this case is in terms of the trigonometric functions. In general, although the problem of determining the lines of force when the motion is plane may be regarded as solved it is possible to obtain their equations only in the very limited number of cases in which the integration can be carried out.

§ 4. Particular cases of plane motion

(a) Rectilinear motion.

Taking the x axis as the axis of motion we see from § 3 that the solution depends on the integration

$$c\int \frac{\zeta'' \, d\tau}{\zeta' \sqrt{c^2 - \zeta'^2}}$$

In this particular case, however, the general solution can be obtained at once from the Riccatian equation which becomes

$$2(c^2 - \zeta'^2)\sigma' = c\zeta''(1-\sigma^2)$$

This gives $\log \frac{1+\sigma}{1-\sigma} = \frac{1}{2} \log \frac{c+v}{c-v} + \log A$ for $v = \zeta'$.

Write $A = \alpha + i\beta$ and we have

$$\frac{1+\sigma}{1-\sigma} = (\alpha + i\beta)\left(\frac{c+v}{c-v}\right)^{1/2}$$

Whence $\sigma = \left\{(\alpha + i\beta)(c^2 - v^2)^{1/2} - (c - v)\right\} \div \left\{(\alpha + i\beta)(c^2 - v^2)^{1/2} + (c - v)\right\}$

From this point we must distinguish between the cases $v \lessgtr c$

Linear motion when $v < c$

$(c^2 - v^2)^{1/2}$ is real and so writing s for the conjugate of σ we have

$$s\left\{(\alpha - \iota\beta)(c^2 - v^2)^{1/2} + (c - v)\right\} = (\alpha - \iota\beta)(c^2 - v^2)^{1/2} - (c - v)$$

The equations giving $l\ m\ n$ in terms of σ and s were seen in §1 to be

$$l = \frac{\sigma + s}{1 + \sigma s} \quad ; \quad im = \frac{\sigma - s}{1 + \sigma s} \quad ; \quad n = \frac{1 - \sigma s}{1 + \sigma s}$$

Accordingly we form the product σs and find

$$\sigma s\left\{(\alpha^2 + \beta^2)(c^2 - v^2) + (c - v)^2 + 2(c - v)(c^2 - v^2)^{1/2}\alpha\right\} = (\alpha^2 + \beta^2)(c^2 - v^2) + (c - v)^2 - 2(c - v)(c^2 - v^2)^{1/2}\alpha$$

Hence writing $(c + v)(\alpha^2 + \beta^2) + (c - v) = R$ we get for the general direction of projection

$$\underline{Rl = (c + v)(\alpha^2 + \beta^2) - (c - v) \ ; \ Rm = 2\beta(c^2 - v^2)^{1/2} \ ; \ Rn = 2\alpha(c^2 - v^2)^{1/2}}$$

α, β are two arbitrary real constants

With these values for l, m, n the lines of force at time t are given parametrically by the equations

$$\underline{x = \xi + c(t - \tau)\, l \quad ; \quad y = c(t - \tau)\, m \quad ; \quad z = c(t - \tau)\, n}$$

It is seen that $\alpha y = \beta z$, or the lines of force lie in planes through the direction of motion. This is, of course, evident à priori from symmetry. If v is constant so also are l m n and we get the known result* that in the case of uniform motion in a right line the line of electric force at any time t through any pt $x\ y\ z$ is got by joining the point x, y, z to the position of the electron at time t.

Owing to the symmetry it is in general sufficient to consider the lines in the plane $z = 0$, so that $\alpha = 0$. For instance in the case of simple harmonic motion we may write

$\xi = a \cos p\tau$ and the lines of electric force in the $x\,y$ plane are

$$x = a \cos p\tau + c(t-\tau)\, l \;; \qquad y = c(t-\tau)\, m$$

where $$l\left\{ \beta^2(c - ap \sin p\tau) + (c + ap \sin p\tau)\right\} = \beta^2(c - ap \sin p\tau) - (c + ap \sin p\tau)$$

$$m\left\{ \beta^2(c - ap \sin p\tau) + (c + ap \sin p\tau)\right\} = 2\beta\left(c^2 - a^2p^2 \sin^2 p\tau\right)^{1/2}$$

and, of course, $l^2 + m^2 = 1$ since $n = 0$.

The single arbitrary constant β gives the single infinity

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* J. J. Thomson Phil. mag. 1881 Vol XI. p. 229

O. Heaviside Phil. mag. 1889 Vol XXVII. p. 324

of lines in the plane xy. To get some idea of the shapes of the lines take the particular line $\beta = 1$

Then $cl = -ap\sin p\tau$; $cm = (c^2 - a^2p^2\sin^2 p\tau)^{1/2}$

Also put $t = 0$ and $\tau = -\tau'$ (for $t > \tau$); then

$$x = a\cos p\tau' + c\tau' l \; ; \; y = c\tau' m.$$

where $cl = ap\sin p\tau'$

$$\text{or} \quad \underline{x = a(\cos p\tau' + p\tau'\sin p\tau') \qquad y = \tau'(c^2 - a^2p^2\sin^2 p\tau')^{1/2}}$$

where τ' can take all positive values. If a is small compared with $p\tau'$ we have $x = a\,p\tau'\sin p\tau'$

whilst $(x' - a\cos p\tau)^2 + y^2 = c^2(t - \tau)^2$ becomes $x^2 + y^2 = c^2\tau'^2$

Thus the curve can be drawn by first drawing the curve $z = a\omega\sin\omega$ which is a sine curve with constantly increasing amplitude. Thus at distances far away from the origin the line of force is a curve of form

$$x = a\tau\sin\tau \qquad x^2 + y^2 = \left(\frac{c}{p}\right)^2\tau^2$$

where τ takes all positive values. This curve is of an oscillatory character with increasing amplitude about the line $x = 0$.

Case of uniformly accelerated motion in a straight line.

Let q be the uniform acceleration and suppose the electron to start from rest at the origin at time $\tau = 0$. Then $\xi = \frac{1}{2} g\tau^2$ $v = g\tau$ so that

$$Rl = \beta^2(c+g\tau) - (c-g\tau) \;; \qquad Rm = 2\beta(c^2-g^2\tau^2)^{1/2}$$

where $R = \beta^2(c+g\tau) + (c-g\tau)$.

It will be sufficient to consider the particular line $\beta = 1$. Then $R = 2c$ and hence $cl = g\tau$ $cm = \beta(c^2-g^2\tau^2)^{1/2}$ so that the equations of the line of force are

$$\underline{x = \tfrac{1}{2} g\tau^2 + (t-\tau)g\tau \qquad ; \quad y = (t-\tau)(c^2-g^2\tau^2)^{1/2}}$$

These equations take a somewhat simpler form if we transfer the origin to the position of the electron at time t; then

$$x = -\tfrac{1}{2} y(t-\tau)^2 \quad ; \quad y = (t-\tau)\{(c-gt)+g(t-\tau)\}^{1/2}\{(c+gt)-g(t-\tau)\}^{1/2}$$

So taking $t-\tau$ as the parameter s we have

$$x = -\tfrac{1}{2} g s^2 \quad ; \quad y^2 = s^2\{c-g(t-s)\}\{c+g(t-s)\}$$

where s takes value from t to zero. On eliminating s this is a quartic curve $\{y^2+4x^2+\frac{2x}{g}(c^2-g^2t^2)\}^2 + 8gt^2x^3 = 0$

It is easy to see now how the lines of force due to a point charge in rectilinear motion are altered in shape when the moving charge is accelerated or retarded. For example imagine the charge to have been moving with uniform velocity for a long time and then to be suddenly affected with a retardation g. We shall take, for convenience, the line $\beta = 1$ which gives $l = \frac{v}{c}$. If at time $\tau = 0$ the electron is at $x = 0$ we have that it is reduced to rest at time $t = \frac{v}{g}$. Let us consider the line of force at this time. The part of the line given by the aggregate of particles imagined projected at times $\tau < 0$ is a straight line which would pass if produced through the point $x = \frac{v^2}{g}$. The line bends at the point where ~~c~~ the line $cl = v$ through the origin meets it and passes through the point $x = \frac{v^2}{2g}$ in a direction perpendicular to the line of motion. As t increases the particles are now all projected from the point $x = \frac{v^2}{2g}$ in a direction perpendicular to $y = 0$ and so at any time t the line of force is made up of three parts — (1) a part which is a straight line (through $x = vt$ if produced) (2) a bent part beginning where this line meets the sphere $x^2 - y^2 + z^2 = c^2t^2$ and ending at $x = \frac{v^2}{2g}$, $y = c\left(t - \frac{v}{g}\right)$

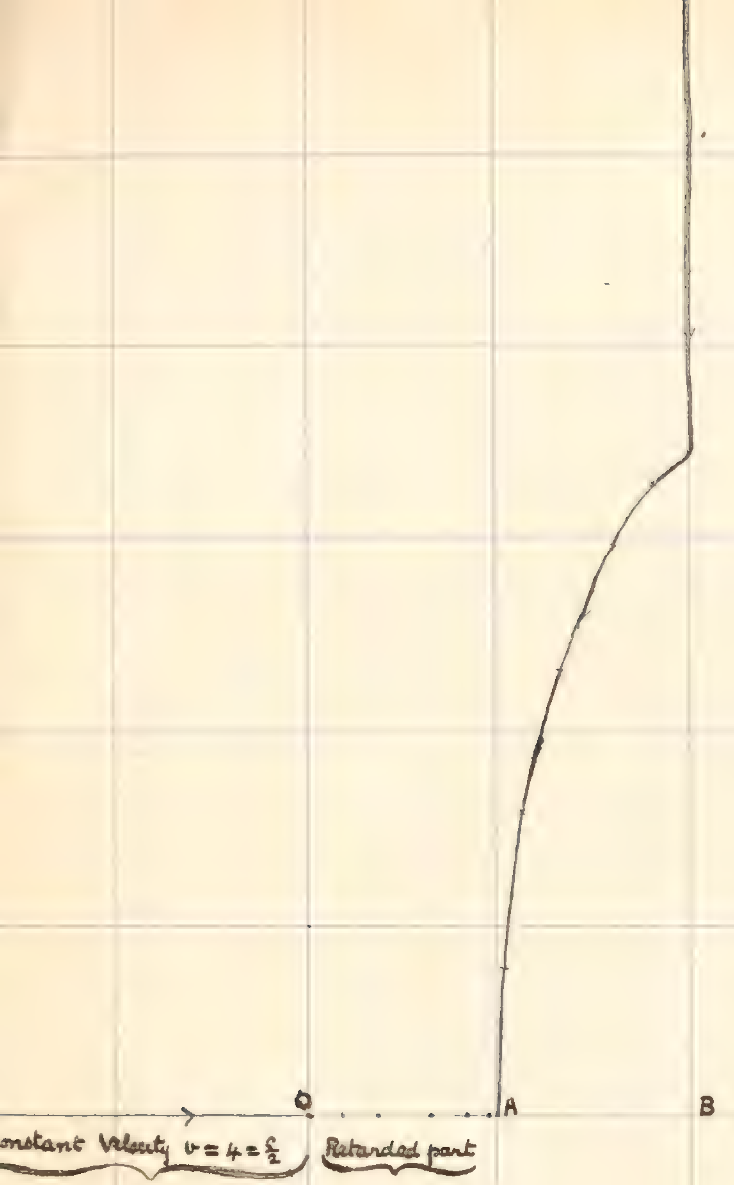
and (3) the part of the line $x = \frac{v^2}{2g}$ from this point to $y=0$. A similar result holds for the lines got from other values of β.

It is evident that, if the electron is stopped instantaneously, we can get the lines of force in the following way. Draw the pencil of lines through the point $x = vt$. The ~~sphere~~ circle with centre at origin and radius $= ct$ will cut these lines in points where the lines of force bend suddenly the remaining portions being the radii of this circle.

This explains how the discontinuity due to the sudden stoppage of the electron spreads out in a spherical wave. The commonly accepted theory of the Rontgen rays is that they are pulses of this type due to the sudden stoppage of quickly moving cathode particles. It is important to notice that the direction of projection at any point of the straight line of motion is determined by the velocity at that point and by that alone.

If we give particular values to v, g, c it is possible to draw the bent portion of the line of force due to the retarded part of the motion. For instance put $v = 4$ $g = c = 8$ Then the electron is reduced to rest in time $\tau = \frac{1}{2}$ its position at that time being $x = v\tau - \frac{1}{2}g\tau^2 = 1$. Putting $\beta = 1$ we get $l = \frac{v}{c}$ Then to get the part due to the accelerated motion we first draw the line at time $t = \frac{1}{2}$ Then if $0 < \tau < \frac{1}{2}$ $l = \frac{4 - 8\tau}{8} = \frac{1-2\tau}{2}$ and $z = 4(\tau - \tau^2)$ The distance from z to the corresponding point on the line of force is $c(t-\tau) = 4(1-2\tau)$ $= 8l$. (See figure overleaf)

Having drawn the line of force at time $t = \frac{1}{2}$ we get the form of the same line at any later time by producing the radii vectores from z to the corresponding point on the line of force a constant distance. (This method of obtaining the subsequent positions of a given line of force follows immediately from the equations of the line of force and is applicable in the general case)


O
A
B
onstant Velocity $v = 4 = \frac{c}{2}$
Retarded part

Form of line of Force due to a uniformly retarded election. This is supp
ave been moving uniformly up to point O with a velocity equal to $\frac{1}{2}$ that of
ght and then to be subjected to a retardation equal to the velocity of light.

e line of force is drawn for the instant at which the electron is reduced
rest. and is that particular line which in the state of uniform motion
perpendicular to the direction of motion

Motion in a straight line when $v > c$

In the equation $\{(\alpha+i\beta)(c^2-v^2)^{1/2}+(c-v)\}\sigma = \{(\alpha+i\beta)(c^2-v^2)^{1/2}-(c-v)\}$

$(c^2-v^2)^{1/2}$ is now a pure imaginary; writing it in form $i(v^2-c^2)^{1/2}$ we see that s, the conjugate of σ is given by the equation

$$\{(\alpha-i\beta)(v^2-c^2)^{1/2}+i(c-v)\}\, s = (\alpha-i\beta)(v^2-c^2)^{1/2}-i(c-v).$$

From these we obtain, as before,

$$\{(\alpha^2+\beta^2)(v+c)+(v-c)\}\, l = (\alpha^2+\beta^2)(v+c) - (v-c)$$

$$\{(\alpha^2+\beta^2)(v+c)+(v-c)\}\, m = 2\alpha(v^2-c^2)^{1/2}$$

$$\{(\alpha^2+\beta^2)(v+c)+(v-c)\}\, n = 2\beta(v^2-c^2)^{1/2}$$

The lines of force lie in planes through the axis of x. Putting $\beta = 0$ we obtain the lines in the plane xy. For these we have $R\,l = \alpha^2(v+c) - (v-c)$; $R\,m = 2\alpha(v^2-c^2)^{1/2}$

where $R = \alpha^2(v+c)+(v-c)$.

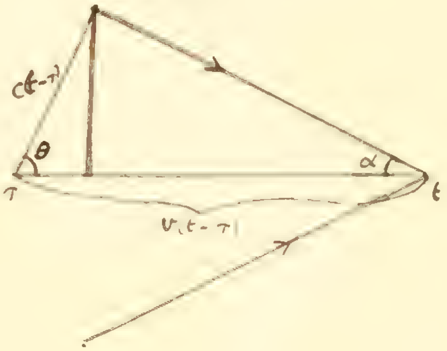
It will be noticed that when $\alpha = 1$ $l = \frac{c}{v}$ and so $\lambda = 0$

Thus in the case of rectilinear motion with constant velocity $v > c$ the generators of the cone $l = \frac{c}{v}$ are directions of projection.

· The cone with vertex at $x = \xi(t)$ and semi vertical $\angle\, \alpha$

where $\sin\alpha = \frac{c}{v}$ ∴ separates the regions where there are lines of electric force from those regions where there are no lines of electric force. The generators of this cone are lines of electric force. This is a known result in the case when v is uniform*

The accompanying diagram will make this clear.



(The lines of force exist only in the left-hand part of the cone)

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* O. Heaviside. Electrical Papers.

Motion in a circle with uniform velocity.

Let p be the constant angular velocity and a the radius of the circle. Then the plane of the circle being the xy plane and the centre of the circle being the origin of coordinates $\xi = a \cos p\tau$; $\eta = a \sin p\tau$, $\zeta = 0$

These give $\phi = a e^{ip\tau}$ and the particular solutions of the Riccatian equation are $\sigma_1 = (i a p e^{ip\tau}) \div c(1+\omega)$ and $\sigma_2 = (i a p e^{ip\tau}) \div c(1-\omega)$

where $c^2\omega^2 = c^2 - v^2 = c^2 - a^2 p^2$.

On substituting these values it is found that the general solution is

$$\sigma = i e^{ip\tau}\left[\frac{c(1+\omega)}{ap} + \frac{1}{(\alpha + i\beta)e^{-ip\tau/\omega} - ap/2c\omega}\right]^{*}$$

If $v < c$ ω is real ; If $v > c$ ω is a pure imaginary so there will be two distinct solutions according as $v \lessgtr c$.

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* The integral $\int \frac{\psi''}{\psi'\omega} d\tau$ of § 3 is in this case $\frac{1}{\omega} \log \psi'$ and

$\psi^* = a e^{-ip\tau}$

Uniform circular motion when $v < c$

Since ω is real the conjugate s of σ is got by merely changing the sign of i wherever it appears. Adding we have

$$\underline{\sigma + s} = \frac{-2c(1+\omega)}{ap} \sin p\tau - 2\,\frac{P \sin p\tau + Q \cos p\tau}{R}$$

where P, Q, R are defined by the equations

$$P = \alpha \cos p\tau/\omega + \beta \sin p\tau/\omega - ap/2c\omega$$

$$Q = \alpha \sin p\tau/\omega - \beta \cos p\tau/\omega$$

$$R = \alpha^2 + \beta^2 + a^2p^2/4c^2\omega^2 - ap/c\omega\left(\alpha \cos p\tau/\omega + \beta \sin p\tau/\omega\right).$$

Similarly on subtracting we obtain

$$\underline{-i(\sigma - s)} = 2c(1+\omega)/ap \,\cos p\tau + 2.\frac{P \cos p\tau - Q \sin p\tau}{R}$$

and on multiplication

$$\underline{\sigma s} = \frac{c^2(1+\omega)^2}{a^2p^2} + \frac{\frac{2c(1+\omega)}{ap}P + 1}{R}$$

From these equations we obtain

$$R(1+\sigma s) = S \quad ; \quad R(1-\sigma s) = T$$

where S and T are defined by the equations

$$S = (\alpha^2+\beta^2)\frac{2}{1-\omega} + \frac{1-\omega}{2\omega^2} - 2\frac{ap}{c\omega}\left(\alpha\cos p\tau,\omega + \beta\sin p\tau,\omega\right)$$

$$T = (\alpha^2+\beta^2)\frac{2\omega}{\omega-1} + \frac{1-\omega}{2\omega}$$

and these values give

$$l \equiv \frac{\sigma+s}{1+\sigma s} = -\frac{ap}{c}\sin p\tau + \frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\sin p\tau - Q\cos p\tau\right\}$$

$$m \equiv -i\frac{\sigma-s}{1+\sigma s} = \frac{ap}{c}\cos p\tau - \frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\cos p\tau + Q\sin p\tau\right\}$$

$$n \equiv \frac{1-\sigma s}{1+\sigma s} = \frac{T}{S}$$

With these values of l m n the parametric equations of the lines of force are $x = a\cos p\tau + c(t-\tau)l$; $y = a\sin p\tau + c(t-\tau)m$; $z = c(t-\tau)n$.

By suitable choice of the constants α, β so as to get particular lines of force these expressions may be considerably simplified. Thus if we put $\alpha = 0 \quad \beta = 0$

$l = -\frac{ap}{c}\sin p\tau$; $m = \frac{ap}{c}\cos p\tau$; $n = \omega$ and the line of force is given by the equations

$$x = a\cos p\tau - ap(t-\tau)\sin p\tau \; ; \; y = a\sin p\tau + ap(t-\tau)\cos p\tau \; ; \; z = c(t-\tau)\omega$$

These equations give $x^2 + y^2 = a^2\{1 + p^2(t-\tau)^2\} = a^2\{1 + z^2 p^2/c^2\omega^2\}$

The line of force lies on the hyperboloid of revolution

$$(x^2+y^2)/a^2 - p^2z^2/c^2\omega^2 = 1.$$

which contains the circle of motion. Writing this in the form $(x/a - pz/c\omega)(x/a + pz/c\omega) = (1 + \frac{y}{a})(1 - y/a)$ we see that the directions of projection are generators of the hyperboloid.

(For the direction of projection at the point $x = a$ $y = 0$, $z = 0$ is

$\frac{x-a}{0} = \frac{y}{ap/c} = \frac{z}{\omega}$ and so is the generator through $x = a$ $y = 0$, $z = 0$, — $x = a$; $y/a - \frac{zp}{c\omega} = 0$).

The form of the line is evident if we consider its projection

on the plane of motion. The equations of the projection are

$x = a\cos p\tau - ap(t-\tau)\sin p\tau$; $y = a\sin p\tau + ap(t-\tau)\cos p\tau$; $z = 0$

giving $$(x - a\cos p\tau)^2 + (y - a\sin p\tau)^2 = a^2p^2(t-\tau)^2$$
$$(x - a\cos p\tau) + \tan p\tau\,(y - a\sin p\tau) = 0$$

These equations are merely those of an involute of the circle $x^2 + y^2 = a^2$. The involute meets the circle at the position of the electron at time $\tau = t$.

To obtain, then, a line of force due to an electron moving with velocity $v < c$ in a circle we unwrap a string which is wound round the ⊙, the tracing point at the end of the string leaving the circle at the point $x = a\cos pt$, $y = a\sin pt$. Having obtained this involute we project it by lines parallel to the axis of z on the hyperboloid $(x^2+y^2)/a^2 - p^2z^2/$[illegible] $= 1$

Lines of force in the plane of motion

To obtain these we put $n = 0$ $\therefore T = 0$ and hence $\alpha^2 + \beta^2 = \left(\frac{1-\omega}{2\omega}\right)^2$ and we may write

$2\omega\alpha = (1-\omega)\cos\theta$; $2\omega\beta = (1-\omega)\sin\theta$ These give

$$\omega^2 S = (1-\omega)K$$

where $K = 1 - ap/c \cos(p\tau/\omega - \theta)$

and finally

$$Kl = \sin p\tau \cos(p\tau/\omega - \theta) - \frac{ap}{c}\sin p\tau - \omega\cos p\tau \sin(p\tau/\omega - \theta)$$

$$Km = -\cos p\tau \cos(p\tau/\omega - \theta) + \frac{ap}{c}\cos p\tau - \omega\sin p\tau \sin(p\tau/\omega - \theta)$$

The single constant θ gives the single infinity of lines of force in the plane of motion.

Putting $\theta = 0$ we get a particular line

$$Kl = \sin p\tau \cos\frac{p\tau}{\omega} - ap/c \sin p\tau - \omega\cos p\tau \sin p\tau/\omega$$

$$Km = -\cos p\tau \cos\frac{p\tau}{\omega} + ap/c \cos p\tau - \omega\sin p\tau \sin\frac{p\tau}{\omega}$$

where the Doppler factor K is now $1 - \frac{ap}{c}\cos\frac{p\tau}{\omega}$.

Circular motion when $v > c$

ω is now a pure imaginary and so it may be written $= \iota\theta$ where θ is real. The functions τ and S are now given by the equations

$$\sigma = \quad i\, e^{\iota pT}\left\{ \frac{c(1+\iota\theta)}{ap} + \frac{1}{(\alpha+\iota\beta)\, e^{-pT/\theta} + \iota ap/2c\theta} \right\}$$

$$s = -i\, e^{-\iota pT}\left\{ \frac{c(1-\iota\theta)}{ap} + \frac{1}{(\alpha-\iota\beta)\, e^{-pT/\theta} - iap/2c\theta} \right\}$$

whence $D(\sigma+s) = \quad -D\left(\frac{2c}{ap}\sin pT + \frac{2c\theta}{ap}\cos pT\right) + 2\, e^{-pT/\theta}\left\{\beta\cos pT - \alpha\sin pT\right\} + \frac{ap}{c\theta}\cos pT$

where $D = \alpha^2 e^{-2pT/\theta} + \left\{\beta\, e^{-pT/\theta} + \frac{ap}{2c\theta}\right\}^2$

Similarly $-\iota D(\sigma - s) = \frac{2c}{ap} D\left\{\cos pT - \theta\sin pT\right\} + 2\, e^{-pT/\theta}\left\{\alpha\cos pT + \beta\sin pT\right\} + \frac{ap}{c\theta}\sin pT$

Further

$$\sigma s = \quad \frac{c^2(1+\theta^2)}{a^2p^2} + \frac{1}{D}\left\{1 + \frac{2c\alpha}{ap} e^{-pT/\theta} - 2c\theta/ap\left(\beta\, e^{-pT/\theta} + ap/2c\theta\right)\right\}$$

$$= \quad 1 + \frac{2c}{apD}\, e^{-pT/\theta}(\alpha - \theta\beta)$$

Hence $1 + \sigma s = 2\left\{1 + \frac{c}{apD} e^{-pT/\theta}(\alpha - \theta\beta)\right\}$

$1 - \sigma s = -\frac{2c}{apD}\, e^{-pT/\theta}(\alpha - \theta\beta).$

Using these values we obtain, after some reduction,

$$l = -\frac{c}{ap}\left[\sin p\tau + \frac{\theta\left\{F\cos p\tau + \frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\sin p\tau\right\}}{G}\right]$$

$$m = \frac{c}{ap}\left[\cos p\tau - \frac{\theta\left\{F\sin p\tau - \frac{c}{ap}e^{-\frac{p\tau}{\theta}}(\theta\alpha+\beta)\cos p\tau\right\}}{G}\right]$$

$$n = \frac{c}{ap}\,e^{-p\tau/\theta}\,\frac{(\theta\beta-\alpha)}{G}$$

where $F = (\alpha^2+\beta^2)e^{-2p\tau/\theta} - a^2p^2/4c^2\theta^2$

and $G = (\alpha^2+\beta^2)e^{-2p\tau/\theta} + \frac{a^2p^2}{4c^2\theta^2} + \frac{c}{ap}e^{-p\tau/\theta}\left(\alpha+\frac{\beta}{\theta}\right)$.

The lines in the plane of motion are obtained by writing $\alpha = \theta\beta$ in these formulae.

Further on writing $\lambda = 0$ we find $\theta\alpha+\beta = 0$; and with this relation between the constants we find

$l = \left\{\frac{c\theta}{ap}\cos p\tau\right\}\frac{1-x}{1+x} - \frac{c}{ap}\sin p\tau$; $m = \left\{\frac{c\theta}{ap}\sin p\tau\right\}\frac{1-x}{1+x} + \frac{c}{ap}\cos p\tau$

$n = -\frac{4c}{ap}\,\frac{\alpha\theta^2 e^{-p\tau/\theta}}{1+x}$ where $x = 4\alpha^2\theta^2 e^{-2p\tau/\theta}$

Thus there are a single infinity of lines which satisfy $\lambda = 0$. Putting $\alpha = 0$ we get that one of these which lies

in the plane of motion. For this line

$$l = -\frac{c}{ap}\left[\sin p\tau - \theta\cos p\tau\right]; \quad m = \frac{c}{ap}\left[\cos p\tau + \theta\sin p\tau\right]; \quad n = 0$$

Write $\sin\phi = \frac{c}{ap}$; Then since $c^2\theta^2 = a^2p^2 - c^2$ we have $\cos\phi = \frac{c\theta}{ap}$ and hence $l = \cos(p\tau+\phi)$; $m = \sin(p\tau+\phi)$

The equations of the line of force are $x = a\cos p\tau + c(t-\tau)\cos(p\tau+\phi)$

$$y = a\sin p\tau + c(t-\tau)\sin(p\tau+\phi).$$

The direction of projection makes a fixed angle with the radius from origin to point of projection so that the line of force winds round the circle in a manner very similar to the involutes of the circle.

Motion in a helix with uniform velocity

It has been found possible to solve the problem in one case where the motion is not in one plane. If the electron moves in a helix with constant

velocity so that $\xi = a \cos p\tau$; $\eta = a \sin p\tau$; $\zeta = d\tau$ where d is a constant the Riccatian equation of §1 may be written

$$2(c^2 - a^2p^2 - d^2)\frac{d\sigma}{d\tau} = ap^2(d-c)e^{ip\tau} - 2i\sigma a^2p^3 + ap^2\sigma^2 e^{-ip\tau}(c+d);$$

making use of the trial solution $\sigma = Ae^{ip\tau}$ we find that

σ_1, σ_2 are particular solutions where

$$(c+d)(1+\omega)\sigma_1 = iape^{ip\tau} \quad ; \quad (c+d)(1-\omega)\sigma_2 = iape^{ip\tau}$$

$$\text{and } (c^2-d^2)\omega^2 = c^2 - d^2 - a^2p^2.$$

Proceeding in the same way as before the general solution is found to be

$$\sigma = ie^{ip\tau}\left\{\frac{(c-d)}{ap}(1+\omega) + \frac{1}{(\alpha+i\beta)e^{-ip\tau/\omega} - \frac{ap}{2(c-d)\omega}}\right\}$$

From this on the work is the same as in the case of circular motion. In the case when $c > d$ and $c^2 < a^2p^2 + d^2$ (so that ω is a pure imaginary) it is found as before that if we write $\omega = i\theta$ and proceed with the solution that $\lambda = 0$ if $\alpha\theta + \beta = 0$. Thus, again, there exist a single infinity of lines of force for which the directions of projection satisfy the equation $\lambda = 0$. It has not been thought necessary to give the results here as they are not essentially different from those

already developed for the case of circular motion.

§ 5.

It was thought that it might be possible to find several curves with the same directions of projection. In other words that, knowing a solution l, m, n of the equations of §1 for a definite curve of motion it might be possible to find another curve or system of curves crossing these lines and such that the lines are directions of projection for them. It was found that this is not in general the case.

Let $\xi(\tau) \cdot \ \eta(\tau) \ \zeta(\tau)$ be the coordinates of a point on the curve for which a solution (l, m, n) is known. Let $\xi_1 \ \eta_1 \ \zeta_1$ be the coordinates of any point on the line $(l \ m \ n)$ through ξ, η, ζ and imagine the electron in the new path to be at the point ξ_1, η_1, ζ_1 at time τ_1. Then $\xi_1 = \xi + c(\tau_1 - \tau) l$ (for if the projected particle is to give a point on a line of force for the new curve it must be at ξ_1, η_1, ζ_1 when the election is there i,e at time τ_1.) The velocity in the new path is $v_1 = \left(\frac{\partial \xi_1}{\partial \tau_1}, \frac{\partial \eta_1}{\partial \tau_1}, \frac{\partial \zeta_1}{\partial \tau_1}\right)$. We shall consider τ as the independent

variable so that $\xi, \eta, \zeta, l, m, n, \xi_1, \eta_1, \zeta_1, \tau_1$ are all functions of the single variable τ; From the equation $\xi_1 = \xi + c(\tau_1 - \tau) l$ we reduce on differentiation $\frac{\partial \xi_1}{\partial \tau_1} - cl = \left(\frac{\partial \xi}{\partial \tau} - cl\right)\frac{\partial \tau}{\partial \tau_1} + c(\tau_1 - \tau)\frac{\partial l}{\partial \tau_1}$

Writing, as in § 1, $\lambda = c - l\xi' - m\eta' - n\zeta'$ and $\lambda_1 = c - l\frac{\partial \xi_1}{\partial \tau_1} - m\frac{\partial \eta_1}{\partial \tau_1} - n\frac{\partial \zeta_1}{\partial \tau_1}$ we get Since $l^2 + m^2 + n^2 = 1$ so that $l\frac{\partial l}{\partial \tau_1} + m\frac{\partial m}{\partial \tau_1} + n\frac{\partial n}{\partial \tau_1} = 0$

$$\lambda_1 = \lambda \frac{\partial \tau}{\partial \tau_1}$$

Differentiating the equation for $\frac{\partial \xi_1}{\partial \tau_1}$ we obtain

$$\frac{\partial^2 \xi_1}{\partial \tau_1^2} = c\frac{\partial l}{\partial \tau_1} + (\xi'' - cl')\left(\frac{\partial \tau}{\partial \tau_1}\right)^2 + (\xi' - cl)\frac{\partial^2 \tau}{\partial \tau_1^2} + c\frac{\partial l}{\partial \tau_1}\left(1 - \frac{\partial \tau}{\partial \tau_1}\right) + c(\tau_1 - \tau)\frac{\partial^2 l}{\partial \tau_1^2}$$

(where as before dashes denote differentiations with regard to τ.)

This may be written

$$\frac{\partial^2 \xi_1}{\partial \tau_1^2} = \xi''\left(\frac{\partial \tau}{\partial \tau_1}\right)^2 + (\xi' - cl)\frac{\partial^2 \tau}{\partial \tau_1^2} + c(\tau_1 - \tau)\frac{\partial^2 l}{\partial \tau_1^2} + 2c\frac{\partial l}{\partial \tau_1}\left(1 - \frac{\partial \tau}{\partial \tau_1}\right).$$

This carries with it two companion equations in η_1 and ζ_1; if we write $p = l\xi'' + m\eta'' + n\zeta''$; $p_1 = l\frac{\partial^2 \xi_1}{\partial \tau_1^2} + m\frac{\partial^2 \eta_1}{\partial \tau_1^2} + n\frac{\partial^2 \zeta_1}{\partial \tau_1^2}$ we derive from the equation just given

$$p_1 = p\left(\frac{\partial \tau}{\partial \tau_1}\right)^2 - \lambda\frac{\partial^2 \tau}{\partial \tau_1^2} + c(\tau_1 - \tau)\left(l\frac{\partial^2 l}{\partial \tau_1^2} + m\frac{\partial^2 m}{\partial \tau_1^2} + n\frac{\partial^2 n}{\partial \tau_1^2}\right)$$

Also differentiating the equation $l\frac{\partial l}{\partial \tau_1} + m\frac{\partial n}{\partial \tau_1} + n\frac{\partial n}{\partial \tau_1} = 0$ we get

$$\sum^{1}_{l,m,n} l\frac{\partial^2 l}{\partial \tau_1^2} = -\sum^{1}_{l,m,n}\left(\frac{\partial l}{\partial \tau_1}\right)^2$$

Now if $(l\ m\ n)$ are to be directions

of projection for the new path we must have three equations of the type

$$(c^2 - v_1^2)\frac{\partial l}{\partial T_1} = \lambda_1 \frac{\partial^2 \xi_1}{\partial T_1^2} + \left(\frac{\partial \xi_1}{\partial T_1} - cl\right) p_1$$

Using the equations already given and dividing across by $\frac{\partial T}{\partial T_1}$ this is

$$\frac{\partial l}{\partial T}\left[c^2 - v_1^2 - c(T_1 - T)p_1\right] = \lambda \frac{\partial^2 \xi_1}{\partial T_1^2} + \left(\frac{\partial \xi}{\partial T} - cl\right) p_1$$

Substituting for $\frac{\partial^2 \xi_1}{\partial T_1^2}$ the value given above we get after a slight reduction,

$$\left\{c^2 - v_1^2 - c(T_1 - T)p_1 - 2c\lambda\frac{\partial T}{\partial T_1}\left(1 - \frac{\partial T}{\partial T_1}\right)\right\}\frac{\partial l}{\partial T} = \left(\frac{\partial T}{\partial T_1}\right)^2\{\lambda\xi'' + (\xi' - cl)p\} + c(T_1 - T)\left\{\lambda\frac{\partial^2 l}{\partial T_1^2} + q_1(\xi' - cl)\right\}$$

where $q_1 = l\frac{\partial^2 l}{\partial T_1^2} + m\frac{\partial^2 m}{\partial T_1^2} + n\frac{\partial^2 n}{\partial T_1^2} = -\left\{\left(\frac{\partial l}{\partial T_1}\right)^2 + \left(\frac{\partial m}{\partial T_1}\right)^2 + \left(\frac{\partial n}{\partial T_1}\right)^2\right\}$

Now squaring and adding the three equations of type

$$\frac{\partial \xi_1}{\partial T_1} - cl = (\xi' - cl)\frac{\partial T}{\partial T_1} + c(T_1 - T)\frac{\partial l}{\partial T_1}$$ we obtain

$$v_1^2 + 2c\lambda_1 - c^2 = (v^2 + 2c\lambda - c^2)\left(\frac{\partial T}{\partial T_1}\right)^2 - c^2(T_1 - T)^2 q_1 + 2c(T_1 - T)\left(\frac{\partial T}{\partial T_1}\right)^2(l'\xi' + m'\eta' + n'\zeta')$$

We have seen in § 2 that

$$(c^2 - v^2)(l'\xi' + m'\eta' + n'\zeta') = \lambda(\bar{v}\,\bar{v}') + (v^2 + c\lambda - c^2)p$$ and since $\lambda_1 = \lambda\frac{\partial T}{\partial T_1}$

we have

$$c^2 - v_1^2 - 2c\lambda\frac{\partial T}{\partial T_1} + 2c\lambda\left(\frac{\partial T}{\partial T_1}\right)^2 = (c^2 - v^2)\left(\frac{\partial T}{\partial T_1}\right)^2 + c^2(T_1 - T)^2 q_1 - 2c(T_1 - T)\left(\frac{\partial T}{\partial T_1}\right)^2\left\{\frac{\lambda(\bar{v}\,\bar{v}') + cp}{c^2 - v^2} - p\right\}$$

Hence using value given for p_1

$$c^2 - v_1^2 - 2c\lambda\frac{\partial T}{\partial T_1} + 2c\lambda\left(\frac{\partial T}{\partial T_1}\right)^2 - c(T_1 - T)p = (c^2 - v^2)\left(\frac{\partial T}{\partial T_1}\right)^2 + c\lambda(T_1 - T)\frac{\partial^2 T}{\partial T_1^2} - 2c(T_1 - T)\left(\frac{\partial T}{\partial T_1}\right)^2\left\{\frac{\lambda(\bar{v}\,\bar{v}' + cp}{c^2 - v^2} - \frac{p}{2}\right\}$$

Accordingly on comparing this with a previous equation and

dividing across by $c(T_1 - T)$ we must have

$$\lambda \frac{\partial^2 l}{\partial T_1^2} + (\zeta' - cl)\, q_1 = \left[\lambda \frac{\partial^2 T}{\partial T^2} + \left(\frac{\partial T}{\partial T_1}\right)^2 \right\} p - 2\lambda \frac{(\bar{v}\, \bar{v}') + cp}{c^2 - v^2} \left\} \right] \frac{\partial l}{\partial T}$$

again $\frac{\partial l}{\partial T_1} = \frac{\partial l}{\partial T} \frac{\partial T}{\partial T_1}$; so $\frac{\partial^2 l}{\partial T_1^2} = \left(\frac{\partial^2 l}{\partial T^2}\right)\left(\frac{\partial T}{\partial T_1}\right)^2 + \frac{\partial l}{\partial T} \frac{\partial^2 T}{\partial T_1^2}$

and since $q_1 = q\left(\frac{\partial T}{\partial T_1}\right)^2$ where $q = l \frac{\partial^2 l}{\partial T^2} + m \frac{\partial^2 m}{\partial T^2} + n \frac{\partial^2 n}{\partial T^2}$

Substituting above for $\frac{\partial^2 l}{\partial T_1^2}$ and q_1 we get

$$\left\{ p - \frac{2\lambda \{(\bar{v}\, \bar{v}') + cp\}}{(c^2 - v^2)} \right\} \frac{\partial l}{\partial T} = (\zeta' - cl) q + \lambda \frac{\partial^2 l}{\partial T^2} \qquad \text{—— (1)}$$

Multiplying the three equations of this type by $\frac{\partial l}{\partial T}$, $\frac{\partial m}{\partial T}$, $\frac{\partial n}{\partial T}$ respectively and adding we get

$$\left\{ p - \frac{2\lambda \{(\bar{v}\, \bar{v}') + cp\}}{c^2 - v^2} \right\} q = -(l'\zeta' + m'\eta' + n\zeta') q + \tfrac{1}{2} \lambda \frac{\partial q}{\partial T}$$

$$= \left\{ p - \frac{\lambda(\bar{v}\, \bar{v}' + cp)}{c^2 - v^2} \right\} q + \tfrac{1}{2} \lambda \frac{\partial q}{\partial T}$$

Whence since $\frac{\partial \lambda}{\partial T} = -\{p + l'\zeta' + m'\eta' + n'\zeta'\} = -\lambda \frac{(\bar{v}\, \bar{v}') + cp}{c^2 - v^2}$

$$\frac{\partial \lambda}{\partial T} \Big/ \lambda = \frac{1}{2} \frac{\partial q}{\partial T} \Big/ q \qquad \text{giving} \quad q = a\lambda^2$$

Hence it is impossible to find a second path unless $l'^2 + m'^2 + n'^2 \propto \lambda^2$

In general it will be impossible to find a new path; to be able to do so the equations marked (1) above must be satisfied.

Biography

I, Francis D. Murnaghan, was born at Omagh, Co. Tyrone, Ireland, on August 4th 1893 the sixth son of George and Angela Murnaghan. I received my early education at the School of the Christian Brothers, in Omagh, and proceeded to University College, Dublin (1910 – 14) where I obtained the degrees of B.A and M.A. I was awarded (1914), by the National University of Ireland, a Traveling Studentship in Mathematics and Mathematical Physics and came to Johns Hopkins in the Autumn of 1914. I was granted a Fellowship by Courtesy for the year 1915 – 16.

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